Improved Generalized Gram-Charlier Expansions based on Multivariate Skew Distributions

Gram-Charlier

Gram-Charlier Expansion

$$a(Y) = f(Y) + \frac{\left(C_2(A) - C_2(F)\right)}{2!} \frac{\partial^2 f(Y)}{\partial Y \partial Y^T} - \frac{\left(C_3(A) - C_3(F)\right)}{3!} \frac{\partial^3 f(Y)}{\partial Y \partial Y^T \partial Y}$$
$$+ \frac{\left(C_4(A) - C_4(F)\right) + 3\left(C_2(A) - C_2(F)\right)^2}{4!} \frac{\partial^4 f(Y)}{\partial Y \partial Y^T \partial Y \partial Y^T} + \varepsilon(Y)$$
$$= f(Y) + \frac{\left(C_2(A) - C_2(F)\right)}{2!} f^{(2)}(Y) - \frac{\left(C_3(A) - C_3(F)\right)}{3!} f^{(3)}(Y)$$
$$+ \frac{\left(C_4(A) - C_4(F)\right) + 3\left(C_2(A) - C_2(F)\right)^2}{4!} f^{(4)}(Y) + \varepsilon(Y)$$

A(Y): True Distribution

F(Y): Approximating Distribution.

C: cumulant

- the cumulants of a probability distribution provide an alternative to the moments of the distribution.
- In some cases theoretical treatments of problems in terms of cumulants have an advantage of simplicity over using moments.
- The cumulant generating function is the logarithm of the moment generating function.
- The first cumulant is the expected value; the second and third cumulants are respectively the second and third central moments (the second central moment is the variance); but the higher cumulants are neither moments nor central moments, but rather more complicated polynomial functions of the moments.

Multivariate Integration

- Adaptive quadrature (package *adapt*)
  - ✓ Adaptive algorithms are now used widely for the numerical calculation of multiple integrals.
  - ✓ The routine operates by repeated subdivision of the hyper-rectangular region into smaller hyper-rectangles.
  - ✓ In each subregion, the integral is estimated using a rule of degree seven, and an error estimate is obtained by comparison with a rule of degree seven which uses a subset of the same points. These subdivisions are designed to dynamically concentrate the computational work in the subregions where the integrand is most irregular, and thus adapt to the behaviour of the integrand. Genz (1991) gives a detailed description in the context of adaptive numerical integration for simplices.
  - ✓ Genz (1992) suggested that such subregion adaptive integration algorithms can be used effectively in some multiple integration problems arising in statistics.
  - ✓ The key to good solutions for these problems is the choice of an appropriate transformation

from the infinite integration region for the original problem to a suitable finite region for the subregion adaptive algorithm.

- ✓ Generally, adaptive algorithms are just as efficient and effective as traditional algorithms for "well-behaved" integrands, but are also effective for "badly-behaved" integrands for which traditional algorithms fail.

- Monte Carlo Methods (Package *R2Cuba*)

✓ The basic MC method iteratively approximates a definite integral by uniformly sampling from the domain of integration, and averaging the function values at the samples.

✓ The integrand is treated as a random variable, and the sampling scheme yields a parameter estimate of the mean, or expected value of the random variable.

✓ This algorithm uses importance sampling as a variance-reduction technique. This algorithm iteratively builds up a piecewise constant weight function, represented on a rectangular grid. Each iteration consists of a sampling step followed by a refinement of the grid.

- Parallel Computing (Package snow)

I. Multivariate Skew Normal Distribution:

$X \to MSN(\mu,\Sigma,\alpha)$, Let $Y = X-\mu$, $Y \to MSN(\Sigma,\alpha)$

◆ n-dimension Multivariate Skew Normal Distribution:

$$f_{SN}(Y;\Sigma,\alpha) = 2 f_n(Y;\Sigma)\Phi(\alpha^T Y)$$

$f_n(\cdot;\Sigma)$ is the pdf of n-dimension multivariate normal distribution and is $\Phi(\cdot)$ is standard normal distribution. Let $N_1(\cdot)$ is the pdf of 1-dimension standard normal distribution

$$\text{where } f_n(Y) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}Y^T\Sigma^{-1}Y\right), \quad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt, \quad N_1(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

- $f_{SN}^{(1)}(Y) = \dfrac{df_{SN}(Y)}{dY} = 2\left(f_n^{(1)}(Y)\Phi(\alpha^T Y) + f_n(Y)N_1(\alpha^T Y)\alpha\right)$
- $f_{SN}^{(2)}(Y) = \dfrac{d^2 f_{SN}(Y)}{dYdY^T} = 2\Big[f_n^{(2)}(Y)\Phi(\alpha^T Y) + f_n^{(1)}(Y)N_1(\alpha^T Y)\alpha^T$

$$+\alpha\left(f_n^{(1)}(Y)\right)^T N_1(\alpha^T Y) + f_n(Y)N_1^{(1)}(\alpha^T Y)\alpha\alpha^T\Big]$$

- $f_{SN}^{(3)}(Y) = \dfrac{d^3 f_{SN}(Y)}{dYdY^T dY}$

$$= 2\Big[f_n^{(3)}(Y)\Phi(\alpha^T Y) + \alpha \otimes f_n^{(2)}(Y) N_1(\alpha^T Y) + vec\big(f_n^{(2)}(Y)\big)\alpha^T N_1(\alpha^T Y)$$
$$+\alpha \otimes f_n^{(1)}(Y)\alpha^T N_1^{(1)}(\alpha^T Y) + \big(f_n^{(2)}(Y)\big)^T \otimes \alpha N_1(\alpha^T Y) + (\alpha\otimes\alpha)\big(f_n^{(1)}(Y)\big)^T N_1^{(1)}(\alpha^T Y)$$
$$+ f_n^{(1)}(Y)\otimes \alpha\alpha^T N_1^{(1)}(\alpha^T Y) + f_n(Y) N_1^{(2)}(\alpha^T Y)\alpha\otimes\alpha\alpha^T \Big]$$

- $f_{SN}^{(4)}(Y) = \dfrac{d^4 f_{SN}(Y)}{dYdY^T dYdY^T}$

$$= 2\Big[f_n^{(4)}(Y)\Phi(\alpha^T Y) + \alpha^T \otimes f_n^{(3)}(Y) N_1(\alpha^T Y) + \alpha \otimes \big(f_n^{(3)}(Y)\big)^T N_1(\alpha^T Y)$$
$$+\alpha\alpha^T \otimes f_n^{(2)}(Y) N_1(\alpha^T Y) + f_n^{(3)}(Y)\otimes \alpha^T N_1(\alpha^T Y) + \alpha^T \otimes vec\big(f_n^{(2)}(Y)\big)\alpha^T N_1^{(1)}(\alpha^T Y)$$
$$+\alpha \otimes f_n^{(2)}(Y)\big(I_N \otimes \alpha^T\big) N_1^{(1)}(\alpha^T Y) + \alpha\alpha^T \otimes f_n^{(1)}(Y)\alpha^T N_1^{(2)}(\alpha^T Y)$$
$$+\big(f_n^{(3)}(Y)\big)^T \otimes \alpha N_1(\alpha^T Y) + \alpha^T \otimes \big(f_n^{(2)}(Y)\big)^T \otimes \alpha N_1^{(1)}(\alpha^T Y)$$
$$+(\alpha\otimes\alpha)\Big(vec\big(f_n^{(2)}(Y)\big)\Big)^T N_1^{(1)}(\alpha^T Y) + \big(\alpha\alpha^T \otimes \alpha\big)\otimes\big(f_n^{(1)}(Y)\big)^T N_1^{(2)}(\alpha^T Y)$$
$$+ f_n^{(2)}(Y)\otimes \alpha\alpha^T N_1^{(1)}(\alpha^T Y) + f_n^{(1)}(Y)\otimes\alpha^T \otimes \alpha\alpha^T N_1^{(2)}(\alpha^T Y)$$
$$+\big(f_n^{(1)}(Y)\big)^T \otimes \alpha \otimes \alpha\alpha^T N_1^{(2)}(\alpha^T Y) + f_n(Y) N_1^{(3)}(\alpha^T Y)\alpha\alpha^T \otimes \alpha\alpha^T \Big]$$

◆ Cumulants of Multivariate Skew Normal Distribution

- Let $\delta = \dfrac{\Sigma\alpha}{\left(1+\alpha^T\Sigma\alpha\right)^{1/2}}$

$$C_1 = \sqrt{\frac{2}{\pi}}\delta,\quad C_2 = \Sigma - \frac{2}{\pi}\delta\delta^T,\quad C_3 = \sqrt{\frac{2}{\pi}}\left(\frac{4}{\pi}-1\right)\delta^{\otimes 2}\delta^T,\quad C_4 = \frac{4}{\pi}\left(2-\frac{6}{\pi}\right)\delta^{\otimes 2}\left(\delta^{\otimes 2}\right)^T$$

◆ Statistical Cumulants:

Moments: $m_k[X] = E\left[X\left(X^T\right)^{\otimes k-1}\right], \qquad k = 1, 2, \cdots$

Central Moments: $\bar{m}_k[X] = E\left[\left(X - E[X]\right)\left(\left(X - E[X]\right)^T\right)^{\otimes k-1}\right], \qquad k = 1, 2, \cdots$

$$C_1 = m_1[X]$$
$$C_2 = \bar{m}_2[X]$$
$$C_3 = \bar{m}_3[X] = m_3[X] - m_2[X]\otimes E[X]^T - E[X]^T \otimes m_2[X]$$
$$-E[X]\big(vec(m_2[X])\big)^T + 2E[X]\left(E[X]^T\right)^{\otimes 2}$$
$$C_4 = \bar{m}_4[X] - \bar{m}_2[X]\otimes\big(vec(\bar{m}_2[X])\big)^T - \Big[\big(vec(\bar{m}_2[X])\big)^T \otimes \bar{m}_2[X]\Big]\Big[I_n + I_n \otimes K_{n,n}\Big]$$

Where $K_{n,n}$ is a commutation matrix

II. Multivariate Skew t Distribution:

$X \to MST_v(\mu,\Sigma,\alpha,v)$, Let $Y = X-\mu$, $Y \to MSN(\Sigma,\alpha,v)$, $X = Z\big/\sqrt{W/v}$

- n-dimension Multivariate Skew t Distribution:

$$f_v(Y;\Sigma,\alpha,v) = 2t_n(Y;\Sigma,v)T_1\left(\alpha^T Y\left(\frac{v+n}{Y^T\Sigma^{-1}Y+v}\right)^{1/2};v+n\right)$$

$t_n(\cdot;\Sigma,v)$ is the pdf of n-dimension multivariate t distribution and is $T_1(\cdot;p)$ is student t distribution with p degree of freedom. Let $t_1(\cdot)$ is the pdf of 1-dimension student t distribution

$$\text{where } t_n(Y;\Sigma,v) = \frac{\Gamma\left(\frac{v+n}{2}\right)}{(\pi v)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}\Gamma\left(\frac{v}{2}\right)}\left[1+\frac{Y^T\Sigma^{-1}T}{v}\right]^{-\frac{(v+n)}{2}},$$

$$T_1(x;p) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi p}\,\Gamma\left(\frac{p}{2}\right)}\int_{-\infty}^{x}\left(1+\frac{t^2}{p}\right)^{-\left(\frac{p+1}{2}\right)}dt,\quad t_1(x;p) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi p}\,\Gamma\left(\frac{p}{2}\right)}\left(1+\frac{x^2}{p}\right)^{-\left(\frac{p+1}{2}\right)}$$

$$\text{Let } g(Y) = \alpha^T Y\left(\frac{v+n}{Y^T\Sigma^{-1}Y+v}\right)^{1/2}$$

- $f_v^{(1)}(Y) = 2\left[t_n^{(1)}(y;v)T_1(g(Y);v+n)+t_n(y;v)t_1(g(Y);v+n)g^{(1)}(Y)\right]$
- $$f_v^{(2)}(Y) = 2\Big\{t_n^{(2)}(y;v)T_1(g(Y);v+n)+t_n^{(1)}(y;v)t_1(g(Y);v+n)\left[g^{(1)}(Y)\right]^T$$
$$+\left[t_n^{(1)}(y;v)\right]^T\otimes t_1(g(Y);v+n)g^{(1)}(Y)+t_n(y;v)t_1^{(1)}(g(Y);v+n)g^{(1)}(Y)\left[g^{(1)}(Y)\right]^T$$
$$+t_n(y;v)t_1(g(Y);v+n)g^{(2)}(Y)\Big\}$$

- $$f_v^{(3)}(Y) = 2\Big\{ t_n^{(3)}(y;v) T_1\big(g(Y);v+n\big) + \big[g^{(1)}(Y)\big] \otimes t_n^{(2)}(y;v) t_1\big(g(Y);v+n\big)$$
$$+ vec\big(t_n^{(2)}(y;v)\big) t_1\big(g(Y);v+n\big)\big[g^{(1)}(Y)\big]^T + \big[g^{(2)}(Y)\big]^T \otimes t_n^{(1)}(y;v) t_1\big(g(Y);v+n\big)$$
$$+ \Big[\big[g^{(1)}(Y)\big]\big[g^{(1)}(Y)\big]^T\Big] \otimes t_n^{(1)}(y;v) t_1^{(1)}\big(g(Y);v+n\big)$$
$$+ \big[t_n^{(2)}(y;v)\big]^T \otimes t_1\big(g(Y);v+n\big) g^{(1)}(Y) + \big[t_n^{(1)}(y;v)\big]^T \otimes t_1\big(g(Y);v+n\big) vec\big(g^{(2)}(Y)\big)$$
$$+ \Big[\big[g^{(1)}(Y)\big] \otimes \big[g^{(1)}(Y)\big]\Big]\big[t_n^{(1)}(y;v)\big]^T t_1^{(1)}\big(g(Y);v+n\big)$$
$$+ t_n^{(1)}(y;v) \otimes t_1^{(1)}\big(g(Y);v+n\big) g^{(1)}(Y)\big[g^{(1)}(Y)\big]^T$$
$$+ t_n(y;v) t_1^{(2)}\big(g(Y);v+n\big) g^{(1)}(Y) \otimes g^{(1)}(Y)\big[g^{(1)}(Y)\big]^T$$
$$+ t_n(y;v) t_1^{(1)}\big(g(Y);v+n\big) vec\big(g^{(2)}(Y)\big)\big[g^{(1)}(Y)\big]^T$$
$$+ t_n(y;v) t_1^{(1)}\big(g(Y);v+n\big)\big[g^{(2)}(Y)\big]^T \otimes g^{(1)}(Y)$$
$$+ t_n^{(1)}(y;v) \otimes t_1\big(g(Y);v+n\big) g^{(2)}(Y) + t_n(y;v) t_1^{(1)}\big(g(Y);v+n\big) g^{(1)}(Y) \otimes g^{(2)}(Y)$$
$$+ t_n(y;v) t_1\big(g(Y);v+n\big) g^{(3)}(Y)\Big\}$$

- $$f_v^{(4)}(Y) = 2\Big\{t_n^{(4)}(y;v)T_1\left(g(Y);v+n\right) + \left[g^{(1)}(Y)\right]^T \otimes t_n^{(3)}(y;v)t_1\left(g(Y);v+n\right)$$

$$+\left[g^{(1)}(Y)\right]\otimes\left[t_n^{(3)}(y;v)\right]^T t_1\left(g(Y);v+n\right)+\left[g^{(2)}(Y)\right]\otimes t_n^{(2)}(y;v)t_1\left(g(Y);v+n\right)$$

$$+\left[g^{(1)}(Y)\right]\left[g^{(1)}(Y)\right]^T \otimes t_n^{(2)}(y;v)t_1^{(1)}\left(g(Y);v+n\right)$$

$$+t_n^{(3)}(y;v)\otimes t_1\left(g(Y);v+n\right)\left[g^{(1)}(Y)\right]^T + vec\left(t_n^{(2)}(y;v)\right)\otimes t_1\left(g(Y);v+n\right)\left[vec\left(g^{(2)}(Y)\right)\right]^T$$

$$+vec\left(t_n^{(2)}(y;v)\right)\otimes t_1^{(1)}\left(g(Y);v+n\right)\left[g^{(1)}(Y)\right]^T\otimes\left[g^{(1)}(Y)\right]^T$$

$$+\left[g^{(2)}(Y)\right]^T\otimes t_n^{(2)}(y;v)K_{2,2}t_1\left(g(Y);v+n\right)+\left[g^{(3)}(Y)\right]^T\otimes t_n^{(1)}(y;v)t_1\left(g(Y);v+n\right)$$

$$+\left[g^{(1)}(Y)\right]^T\otimes\left[g^{(2)}(Y)\right]^T\otimes t_n^{(1)}(y;v)t_1^{(1)}\left(g(Y);v+n\right)$$

$$+\left[\left[g^{(1)}(Y)\right]\left[g^{(1)}(Y)\right]^T\right]\otimes t_n^{(2)}(y;v)K_{2,2}t_1^{(1)}\left(g(Y);v+n\right)$$

$$+\left[\left[g^{(1)}(Y)\right]\left[vec\left(g^{(2)}(Y)\right)\right]^T\right]\otimes t_n^{(1)}(y;v)t_1^{(1)}\left(g(Y);v+n\right)$$

$$+\left[\left[g^{(2)}(Y)\right]\otimes\left[g^{(1)}(Y)\right]^T\right]\otimes t_n^{(1)}(y;v)t_1^{(1)}\left(g(Y);v+n\right)$$

$$+\left[\left[g^{(1)}(Y)\right]^T\otimes\left[g^{(1)}(Y)\right]\left[g^{(1)}(Y)\right]^T\right]\otimes t_n^{(1)}(y;v)t_1^{(2)}\left(g(Y);v+n\right)$$

$$+\left[t_n^{(3)}(y;v)\right]^T\otimes t_1\left(g(Y);v+n\right)g^{(1)}(Y)+\left\{\left[t_n^{(2)}(y;v)\right]^T\otimes g^{(2)}(Y)\right\}K_{2,2}t_1\left(g(Y);v+n\right)$$

$$+\left\{\left[t_n^{(2)}(y;v)\right]^T\otimes g^{(1)}(Y)\left[g^{(1)}(Y)\right]^T\right\}K_{2,2}t_1^{(1)}\left(g(Y);v+n\right)$$

$$+vec\left(g^{(2)}(Y)\right)\left[vec\left(t_n^{(2)}(y;v)\right)\right]^T t_1\left(g(Y);v+n\right)+g^{(3)}(Y)\otimes\left[t_n^{(1)}(y;v)\right]^T t_1\left(g(Y);v+n\right)$$

$$+\left\{vec\left(g^{(2)}(Y)\right)\left[g^{(1)}(Y)\right]^T\right\}\otimes\left[t_n^{(1)}(y;v)\right]^T t_1^{(1)}\left(g(Y);v+n\right)$$

$$+\left[\left[g^{(1)}(Y)\right]\otimes\left[g^{(1)}(Y)\right]\right]\left[vec\left(t_n^{(2)}(y;v)\right)\right]^T t_1^{(1)}\left(g(Y);v+n\right)$$

$$+\left\{\left[t_n^{(1)}(y;v)\right]^T\otimes\left[\left[g^{(1)}(Y)\right]\otimes\left[g^{(2)}(Y)\right]\right]\right\}K_{2,2}t_1^{(1)}\left(g(Y);v+n\right)$$

$$+\left[\left[g^{(2)}(Y)\right]\otimes\left[g^{(1)}(Y)\right]\right]\otimes\left[t_n^{(1)}(y;v)\right]^T t_1^{(1)}\left(g(Y);v+n\right)$$

$$+\left[\left[g^{(1)}(Y)\right]\left[g^{(1)}(Y)\right]^T\otimes\left[g^{(1)}(Y)\right]\right]\otimes\left[t_n^{(1)}(y;v)\right]^T t_1^{(2)}\left(g(Y);v+n\right)$$

$$+t_n^{(2)}(y;v)\otimes t_1^{(1)}\left(g(Y);v+n\right)g^{(1)}(Y)\left[g^{(1)}(Y)\right]^T$$

$$+t_n^{(1)}(y;v)\otimes t_1^{(1)}\left(g(Y);v+n\right)g^{(2)}(Y)\otimes\left[g^{(1)}(Y)\right]^T$$

$$+t_n^{(1)}(y;v)\otimes t_1^{(1)}\left(g(Y);v+n\right)g^{(1)}(Y)\left[vec\left(g^{(2)}(Y)\right)\right]^T$$

$$+t_n^{(1)}(y;v)\otimes t_1^{(2)}\left(g(Y);v+n\right)\left[g^{(1)}(Y)\right]^T\otimes g^{(1)}(Y)\left[g^{(1)}(Y)\right]^T+\cdots$$

$$
\begin{aligned}
f_v^{(4)}(Y) = \cdots &+ t_n(y;v)t_1^{(2)}(g(Y);v+n)\left\{g^{(1)}(Y)\left[g^{(1)}(Y)\right]^T \otimes g^{(2)}(Y)\right\}K_{2,2} \\
&+t_n(y;v)t_1^{(2)}(g(Y);v+n)\left\{g^{(1)}(Y)\otimes\left[g^{(1)}(Y)\right]\right\}vec\left(g^{(2)}(Y)\right) \\
&+t_n(y;v)t_1^{(2)}(g(Y);v+n)\left[g^{(2)}(Y)\right]\otimes\left\{g^{(1)}(Y)\left[g^{(1)}(Y)\right]^T\right\} \\
&+\left[t_n^{(1)}(y;v)\right]^T \otimes t_1^{(2)}(g(Y);v+n)g^{(1)}(Y)\otimes g^{(1)}(Y)\left[g^{(1)}(Y)\right]^T \\
&+t_n(y;v)t_1^{(3)}(g(Y);v+n)g^{(1)}(Y)\left[g^{(1)}(Y)\right]^T \otimes g^{(1)}(Y)\left[g^{(1)}(Y)\right]^T \\
&+t_n(y;v)t_1^{(1)}(g(Y);v+n)g^{(3)}(Y)\otimes\left[g^{(1)}(Y)\right]^T \\
&+t_n(y;v)t_1^{(1)}(g(Y);v+n)vec\left(g^{(2)}(Y)\right)\left[vec\left(g^{(2)}(Y)\right)\right]^T \\
&+\left[t_n^{(1)}(y;v)\right]^T \otimes t_1^{(1)}(g(Y);v+n)\left\{vec\left(g^{(2)}(Y)\right)\left[g^{(1)}(Y)\right]^T\right\} \\
&+t_n(y;v)t_1^{(2)}(g(Y);v+n)vec\left(g^{(2)}(Y)\right)\left\{\left[g^{(1)}(Y)\right]^T \otimes\left[g^{(1)}(Y)\right]^T\right\} \\
&+t_n(y;v)t_1^{(1)}(g(Y);v+n)\left[g^{(3)}(Y)\right]^T \otimes g^{(1)}(Y) \\
&+t_n(y;v)t_1^{(1)}(g(Y);v+n)\left\{\left[g^{(2)}(Y)\right]^T \otimes g^{(2)}(Y)\right\}K_{2,2} \\
&+\left[t_n^{(1)}(y;v)\right]^T \otimes t_1^{(1)}(g(Y);v+n)\left[g^{(2)}(Y)\right]^T \otimes g^{(1)}(Y) \\
&+t_n(y;v)t_1^{(2)}(g(Y);v+n)\left[g^{(1)}(Y)\right]^T \otimes\left[g^{(2)}(Y)\right]^T \otimes g^{(1)}(Y) \\
&+t_n^{(1)}(y;v)\otimes t_1(g(Y);v+n)\left[g^{(3)}(Y)\right]^T + t_n^{(2)}(y;v)\otimes g^{(2)}(Y)t_1(g(Y);v+n) \\
&+\left[g^{(1)}(Y)\right]^T \otimes t_n^{(1)}(y;v)\otimes g^{(2)}(Y)t_1^{(1)}(g(Y);v+n) \\
&+t_n(y;v)t_1^{(1)}(g(Y);v+n)g^{(1)}(Y)\otimes\left[g^{(3)}(Y)\right]^T \\
&+t_n(y;v)t_1^{(1)}(g(Y);v+n)g^{(2)}(Y)\otimes g^{(2)}(Y) \\
&+\left[t_n^{(1)}(y;v)\right]^T \otimes t_1^{(1)}(g(Y);v+n)g^{(1)}(Y)\otimes g^{(2)}(Y) \\
&+t_n(y;v)t_1^{(2)}(g(Y);v+n)\left\{g^{(1)}(Y)\left[g^{(1)}(Y)\right]^T\right\}\otimes g^{(2)}(Y) \\
&+\left[t_n^{(1)}(y;v)\right]^T \otimes t_1(g(Y);v+n)g^{(3)}(Y) + t_n(y;v)t_1^{(1)}(g(Y);v+n)\left[g^{(1)}(Y)\right]^T \otimes g^{(3)}(Y) \\
&+t_n(y;v)t_1(g(Y);v+n)g^{(4)}(Y)\Big\}
\end{aligned}
$$

- Central Moments of Multivariate Skew t Distribution

$$M_1 = c\delta, \quad \text{where } \delta = \frac{\Sigma\alpha}{\left(1+\alpha^T\Sigma\alpha\right)}, \quad c = \frac{\frac{v}{\pi}\Gamma\left(\frac{v-1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)}$$

$$M_2 = \left(\frac{v}{v-2}\right)\Sigma$$

$$M_3 = \left(\frac{cv}{v-3}\right)\left\{\delta\otimes\Sigma + vec(\Sigma)\delta^T + \left(I_n\otimes\delta\right)\Sigma - \delta\otimes\delta^T\otimes\delta\right\}$$

$$M_4 = \left(\frac{v^2}{(v-2)(v-4)}\right)\left\{\left(I_{n^2} + K_{n.n}\right)\left(\Sigma\otimes\Sigma\right) + vec(\Sigma)\left(vec(\Sigma)\right)^T\right\}$$

- Cumulants of Multivariate Skew t Distribution

$$C_2 = M_2$$
$$C_3 = M_3$$
$$C_4 = M_4 - 3M_2^{\otimes 2}$$

- Statistical Cumulants:

Moments: $m_k[X] = E\left[X\left(X^T\right)^{\otimes k-1}\right], \qquad k = 1, 2, \cdots$

Central Moments: $\bar{m}_k[X] = E\left[\left(X - E[X]\right)\left(\left(X - E[X]\right)^T\right)^{\otimes k-1}\right], \qquad k = 1, 2, \cdots$

$$C_1 = m_1[X]$$
$$C_2 = \bar{m}_2[X]$$
$$C_3 = \bar{m}_3[X] = m_3[X] - m_2[X]\otimes E[X]^T - E[X]^T\otimes m_2[X]$$
$$-E[X]\left(vec\left(m_2[X]\right)\right)^T + 2E[X]\left(E[X]^T\right)^{\otimes 2}$$
$$C_4 = \bar{m}_4[X] - \bar{m}_2[X]\otimes\left(vec\left(\bar{m}_2[X]\right)\right)^T - \left[\left(vec\left(\bar{m}_2[X]\right)\right)^T\otimes\bar{m}_2[X]\right]\left[I_n + I_n\otimes K_{n,n}\right]$$

Where $K_{n,n}$ is a commutation matrix

III. Multivariate Skew Laplace Distribution:

- n-dimensional multivariate skew Laplace distribution: $X \to MSL_n(\mu, \Sigma, \alpha)$

  α is the skewness parameter

  $$f_{MSL}(x;\mu,\Sigma,\alpha) = \frac{|\Sigma|^{-1/2}}{2^n \pi^{(n-1)/2} \gamma \Gamma\left(\frac{n+1}{2}\right)} \exp\left[-\gamma\sqrt{(x-\mu)^T \Sigma^{-1}(x-\mu)} + (x-\mu)^T \Sigma^{-1}\alpha\right]$$

  where $\gamma = \sqrt{1+\alpha^T\Sigma^{-1}\alpha}$

- $$f_{MSL}(x;\mu,\Sigma,\alpha) = \frac{|\Sigma|^{-1/2}}{2^n \pi^{(n-1)/2} \gamma \Gamma\left(\frac{n+1}{2}\right)} \exp\left[-\gamma\sqrt{(x-\mu)^T \Sigma^{-1}(x-\mu)} + (x-\mu)^T \Sigma^{-1}\alpha\right]$$
  $$= B\exp\left(-\gamma\sqrt{Y^T\Sigma^{-1}Y} + Y^T\Sigma^{-1}\alpha\right) = B\exp(g(Y))$$

  Let $B = \frac{|\Sigma|^{-1/2}}{2^n \pi^{(n-1)/2} \gamma \Gamma\left(\frac{n+1}{2}\right)}$, $\quad g(Y) = -\gamma\sqrt{Y^T\Sigma^{-1}Y} + Y^T\Sigma^{-1}\alpha$

- $f_{MSL}(Y;\Sigma,\alpha) = Be^{g(Y)}$
- $f_{MSL}^{(1)}(Y;\Sigma,\alpha) = Be^{g(Y)}g^{(1)}(Y) = f_{MSL}(Y)g^{(1)}(Y)$
- $$f_{MSL}^{(2)}(Y;\Sigma,\alpha) = \left[f_{MSL}^{(1)}(Y)\right]^T \otimes g^{(1)}(Y) + f_{MSL}(Y)g^{(2)}(Y)$$
  $$= f_{MSL}(Y)\left[g^{(1)}(Y)\right]^T \otimes g^{(1)}(Y) + f_{MSL}(Y)g^{(2)}(Y)$$
- $$f_{MSL}^{(3)}(Y;\Sigma,\alpha) = \left[f_{MSL}^{(2)}(Y)\right]^T \otimes g^{(1)}(Y) + \left[f_{MSL}^{(1)}(Y)\right]^T \otimes vec\left(g^{(2)}(Y)\right)$$
  $$+ f_{MSL}^{(1)}(Y) \otimes g^{(2)}(Y) + f_{MSL}(Y)g^{(3)}(Y)$$
- $$f_{MSL}^{(4)}(Y;\Sigma,\alpha) = \left[f_{MSL}^{(3)}(Y)\right]^T \otimes g^{(1)}(Y) + \left(\left[f_{MSL}^{(2)}(Y)\right]^T \otimes g^{(2)}(Y)\right)K_{2,2}$$
  $$+\left[vec\left(f_{MSL}^{(2)}(Y)\right)\right]^T \otimes vec\left(g^{(2)}(Y)\right) + g^{(3)}(Y) \otimes \left[f_{MSL}^{(1)}(Y)\right]^T$$
  $$+ f_{MSL}^{(2)}(Y) \otimes g^{(2)}(Y) + f_{MSL}^{(1)}(Y) \otimes \left(g^{(3)}(Y)\right)^T$$
  $$+\left[f_{MSL}^{(1)}(Y)\right]^T \otimes g^{(3)}(Y) + f_{MSL}(Y)g^{(4)}(Y)$$

◆ Cumulants of Multivariate Skew Laplace Distribution

$$C_1 = \alpha$$

$$C_2 = \Sigma + \alpha\alpha^T$$

$$C_3 = \left[\alpha \otimes \Sigma + vec(\Sigma)\alpha^T + \Sigma \otimes \alpha\right] + 2\alpha\alpha^T \otimes \alpha$$

$$\begin{aligned} C_4 = {} & 2\alpha^T \otimes \left[\alpha \otimes \Sigma + vec(\Sigma)\alpha^T + \Sigma \otimes \alpha\right] \\ & +2\left\{\Sigma \otimes \alpha^T \otimes \alpha + \left[\alpha \otimes \alpha\right] vec(\Sigma)^T + \left[\alpha\alpha^T \otimes \Sigma\right] K_{n,n}\right\} \\ & +\left[\Sigma \otimes \Sigma + vec(\Sigma) vec(\Sigma)^T + (\Sigma \otimes \Sigma) K_{n,n}\right] + 6\alpha\alpha^T \otimes \alpha\alpha^T \end{aligned}$$